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LETTER TO THE EDITOR

Parallel dynamic for an extremely diluted neural network

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Abstract. We consider a symmetric version of the Derrida-Gardner-Zippelius (DGZ) model. It is shown that in the limit of extreme dilution this modification of the DGZ model can be solved exactly. This means that for the evolution of the main overlap we obtain analytic expressions which (in contrast to the DGZ model) constitute a chain of coupled equations.

The symmetric synaptic connections and full connectivity are known as the basic features which allow one to apply the methods of equilibrium statistical mechanics to develop a theory of neural network models for associative memory [1]. On the other hand, both of these constraints are unrealistic from the biologist's point of view. For getting closer to biology several modifications of the Little-Hopfield model have been proposed, including sparsely connected networks with asymmetric connections, see e.g. [2-5].

An important recent achievement in this direction has been the exact analytic solution of the dynamics of a randomly diluted, asymmetric version of the Little-Hopfield model introduced and studied by Derrida, Gardner and Zippelius (DGZ) model [6]. The model consists of a system of N Ising spins $\{s_j = \pm 1\}_{j=1}^{N}$ (binary neurons) living in vertices of the network $\mathcal{N}_N = \{1, 2, \ldots, N\}$ being connected by the bonds

$$J_{ij} = C_{ij} \sum_{p=1}^{M} \xi_i^{(p)} \xi_j^{(p)}.$$

Here

$$\{C_{ij} \in \{0, 1\}\}_{i,j=1}^{N} \qquad (i \neq j)$$

are independent identically distributed random variables (IIDRV) which represent the dilution and the asymmetry. Realizations

$$\{\xi_i^{(p)}\}_{i=1,p=1}^{NM} = \{\boldsymbol{\xi}^{(p)}\}_{p=1}^{M}$$

of the IIDRV

$$\{\xi_i^{(p)} = \pm 1\}_{i=1,p=1}^{NM}$$

with $\Pr\{\xi_i^{(p)} = \pm 1\} = \frac{1}{2}$, correspond to M stored patterns $\{\boldsymbol{\xi}^{(p)}\}_{p=1}^{M}$. If $P_C^{(N)} \equiv \Pr\{C_{ij} = 1, i \neq j\} = C/N$, then in the limit $N \to \infty$ one gets extremely diluted asymmetric network whose parallel (and random sequential) dynamics can be described exactly. As discovered in [6] two conditions are essential for the exact solution.

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The first is high asymmetry of synaptic connections: $\Pr\{C_{ij}=C_{ji}=1\}=C^2/N^2$, but $\Pr\{C_{ij}=1\cap C_{ji}=0\}=(C/N)(1-C/N)$ due to independence of dilution variables for different pairs, e.g., for (i,j) and (j,i). This means that (for $N\to\infty$) the number k of symmetric connections (i.e. number of pairs (i,j), $i<j$, such that $C_{ij}=C_{ji}=1$) has a Poisson distribution: $P_{\text{symm}}(k)=(\lambda^k/k!)\mathrm{e}^{-\lambda}$ with parameter $\lambda=C^2/2$. So, the number of symmetric connections in the infinite configurations $\boldsymbol{C}=\{C_{ij}\}_{i,j=1}^{\infty}$ is finite with Pr = 1. The second is a local tree structure [1, 7] of the graph $G_\infty[\boldsymbol{C}]$ with vertices $\mathcal{N}_\infty$ and the set of edges $\{(i,j)\colon C_{ij}=1;\ i,j\in\mathcal{N}_\infty\}$ in the limit of extreme dilution. This, in particular, means that in the limit $N\to\infty$:

(i) almost all feedback loops are eliminated;

(ii) with Pr = 1 any two neurons have different clusters of ancestors, i.e. they will remain independent because they receive inputs from two trees which have no neurons in common.

The aim of the present paper is to show that parallel dynamics of extremely diluted networks can be treated explicitly even if asymmetry constraints are relaxed (see remark at the end of [6]). We demonstrate how a probabilistic approach to the description of the parallel dynamics (see [8, 9]) can be applied in this case.

We consider a symmetric diluted network with interactions

$$J_{ij}=\frac{C_{ij}^{\text{new}}}{2C}\sum_{p=1}^{M}\xi_i^{(p)}\xi_j^{(p)}$$

where $C_{ij}^{\text{new}}=\max(C_{ij};C_{ji})$ in the limit of extreme dilution. We are interested in the description of evolution of the main overlap

$$m_N^{(q)}(t)=\frac{1}{N}\sum_{j=1}^{N}\xi_j^{(q)}s_j(t)$$

generated by parallel dynamics (for temperature θ):

$$D_t^{(\theta)}: s_i(t)\to s_i(t+1)$$

with

$$\Pr\{s_i(t+1)=\pm1\}=\left[1+\exp\left(-s_i(t+1)\frac{2}{\theta}\sum_{j=1}^{N}J_{ij}s_j(t)\right)\right]^{-1} \tag{1}$$

in the thermodynamic limit $N\to\infty$, when

$$\lim_{N\to\infty} m_N^{(p)}(t=0)=\delta_{p,q}m^{(q)}(t=0) \qquad p=1,2,\ldots,M \tag{2}$$

and getting to the limit $M\to\infty$, $M/2C\to\alpha$ (after $N\to\infty$), cf the recent paper [10].

Note, that restoration of symmetry does not change the local tree structure of the new graph $G_\infty[\boldsymbol{C}^{\text{new}}]$. The only difference is that now $\Pr\{C_{ij}^{\text{new}}=1,\ i\neq j\}=2(C/N)-(C/N)^2$, i.e. for $N\to\infty$ the probability distribution of the number of symmetric connections for each neuron is Poisson with parameter $\lambda=2C$. Now the limit $C\to\infty$ $(M/2C\to\alpha)$ corresponds to the infinite average connectivity of a such prepared sparse network.

To satisfy initial conditions (2) we suppose that configuration $\{s_i(0)\}_{i=1}^{N}$ is random. It is independent of $\{\boldsymbol{\xi}^{(p)}\}_{p(\neq q)=1}^{N}$ but is correlated with $\boldsymbol{\xi}^{(q)}$:

$$\Pr\{s_i(0)\xi_i^{(q)}=\pm1\}=\tfrac{1}{2}[1\pm m^{(q)}(t=0)].$$

For simplicity we start with the parallel dynamics (1) for zero temperature $\theta=0$. Then dynamics (1) gets the form $D_t^{(\theta=0)}$: $s_i(t)\to s_i(t+1)=\operatorname{sign}[\Sigma_{j=1}^{N}J_{ij}s_j(t)]$.

It is clear that for the first step of dynamics ($D^{(\theta=0)}_{t=1}$) the symmetry of the random field $\boldsymbol{C}^{\text{new}}$ is irrelevant. Hence, for this step one gets the result known for the asymmetric case [6]:

$$m^{(q)}(t=1)=\operatorname{erf}\left[\frac{m^{(q)}(t=0)}{\sqrt{\alpha}}\right] \tag{3}$$

where $\operatorname{erf}(z)=\sqrt{2/\pi}\int_0^z \mathrm{d}x\,\exp\{-x^2/2\}$. Equation (3) coincides with the well known Kinzel formula [11] which is a consequence of general arguments [9]. We have to explain them in order to pass to the next step $t=2$ and then to general formulae.

Let $I_i(\boldsymbol{C}^{\text{new}})=\{j\in\mathcal{N}_\infty:\ C^{\text{new}}_{ij}=1\}$ be the set of neurons connected with a fixed neuron i for a configuration $\boldsymbol{C}^{\text{new}}$ and $|I_i(\boldsymbol{C}^{\text{new}})|$ be the number of these neurons. Then, using $D^{(\theta=0)}_{t=1}$, one gets

$$\xi_i^{(q)}s_i(t=1)=\operatorname{sign}\left[\frac{1}{2C}\sum_{j\in I_i}\xi_j^{(q)}s_j(0)+v^{(q)}_{i,I_i}(t=0)\right] \tag{4}$$

where

$$v^{(q)}_{i,I_i}(t)=\frac{1}{2C}\xi_i^{(q)}\sum_{p(\neq)q}^{M}\xi_i^{(p)}\sum_{j\in I_i}\xi_j^{(p)}s_j(t)\equiv\frac{1}{2C}\sum_{p(\neq q)}^{M}\sum_{j\in I_i}\phi^{p,q}_{i,j}(t). \tag{5}$$

As it is mentioned above, $\Pr\{|I_i|=k\}=(\lambda^k/k!)\mathrm{e}^{-\lambda}$ with $\lambda=2C$. Therefore, in the limit $C\to\infty$ by the large deviations arguments [12] we get

$$\Pr\left\{\left|\frac{|I_i(\boldsymbol{C}^{\text{new}})|}{2C}-1\right|\geqslant\varepsilon\right\}\leqslant\mathrm{e}^{-\gamma C} \tag{6}$$

for any $\varepsilon>0$ and some $\gamma>0$. By the initial conditions $\{\phi^{p,q}_{i,j}(t=0)\}_{p(\neq q),j\in I_i}$ is a realization of the sequence of IIDRV with $\boldsymbol{E}(\phi^{p,q}_{i,j}(t=0))=0$ and $\operatorname{Var}(\phi^{p,q}_{i,j}(t=0))=1$. Hence, in the '$\alpha$'-lim: $M\to\infty$, $M/2C\to\alpha$, taking into account (6) and the central limit theorem (CLT), one gets

$$\text{`}\alpha\text{'-lim}\ v^{(q)}_{i,I_i}(t=0)\overset{d}{=}\sqrt{\alpha}\,\mathcal{N}(0,1) \tag{7}$$

where d means convergence in distribution and $\mathcal{N}(a,b)$ is a Gaussian random variable with expectation a and with variance b. The initial conditions and the strong law of large numbers (SLLN) together with (6) yield

$$\text{`}\alpha\text{'-lim}\frac{1}{2C}\sum_{j\in I_i}\xi_j^{(q)}s_j(0)\overset{\Pr=1}{=}m^{(q)}(0). \tag{8}$$

Therefore, combining (4), (7) and (8) one gets that

$$\Pr\{\xi_i^{(q)}s_i(t=1)=\pm1\}=\frac{1}{\sqrt{2\pi}}\int_{\pm m^{(q)}(0)/\sqrt{\alpha}}^{\infty}\mathrm{d}t\,\exp\left\{-\frac{t^2}{2}\right\} \tag{9}$$

which, gives (3) for 'α'-$\lim(1/N)\sum_{j=1}^N\xi_j^{(q)}s_j(t=1)$ by the SLLN. Similarly, to derive formulae for $m^{(q)}(t=2)$ we have to calculate distribution of $\xi_i^{(q)}s_i(t=2)$ or of $v^{(q)}_{i,I_i}(t=1)$, see (4) and (5).

In contrast to the case of directional bonds, for symmetric connections a feedback is essential for $t\geqslant2$. At time $t=2$ the neuron i receives from the environment I_i, the input which is correlated by the state of the same neuron at $t=0$ (echo). Calculations of these correlations for a fully connected network is a complicated problem [9]. For extremely diluted cases it is easier because of a simple (local tree) structure of the graph $G_\infty[\boldsymbol{C}^{\text{new}}]$.

By the local-tree structure of the graph $G_\infty[C^{\text{new}}]$ variables $\{s_j(t=1)\}_{j\in I_i}$ are correlated due to only one common ancestor $s_i(t=0)$. Hence, to calculate a limiting distribution for $v_{i,I_i}^{(q)}(t=1)$ we have to take into account that $\{\phi_{i,j}^{p,q}(t=1)\}_{j\in I_i}$ are correlated:

$$\phi_{i,j}^{p,q}(t=1)=\operatorname{sign}\left[\frac{1}{2C}\,\xi_i^{(q)}s_i(0)+\xi_i^{(q)}\xi_i^{(p)}\xi_j^{(p)}w_j^{p,q}(t=0)\right]. \tag{10}$$

The first term in the argument of sign appears due to the above common ancestor $i\in I_j$ and it is independent of $j\in I_i$. Here

$$w_j^{p,q}(t)=\frac{1}{2C}\,\xi_j^{(q)}\sum_{k\in I_j\setminus\{i\}}\xi_k^{(q)}s_k(t)+\frac{1}{2C}\sum_{k\in I_j\setminus\{i\}}\sum_{f(\neq q)}^{M}\xi_j^{(f)}\xi_k^{(f)}s_k(t)+\frac{1}{2C}\,s_i(t)\sum_{f(\neq p)}\xi_j^{(f)}\xi_i^{(f)}. \tag{11}$$

The second and the last terms in (11) are independent random variables. Distribution of the last term is independent of the realizations of the variables $\{\xi_i^{(f)}\}_{f=1}^{M}$ and $s_i(t=0)$ which are the only possible source of the dependence of variables $w_j^{(p,q)}(t=0)$ for different j. Consequently $\{w_j^{(p,q)}(t=0)\}_{j\in I_i}$ are independent for different j.

By the initial conditions the first term in (11) converges in the 'α'-lim for $t=0$ to $\xi_j^{(q)}m^{(q)}(t=0)$. The sum of the second and the third term, by (5) and (6), (7), converges to the random variables $\sqrt{\alpha}\mathcal{N}(0,1)$ which are, by the above arguments, independent for different j. Thus

$$\text{'}\alpha\text{'-lim } w_j^{p,q}(t=0)\stackrel{d}{=}\xi_j^{(q)}m^{(q)}(t=0)+\sqrt{\alpha}\mathcal{N}(0,1). \tag{12}$$

This means that the density of the probability distribution for (12) has the form

$$f_{w(0)}(x)=\frac{1}{2\sqrt{2\pi\alpha}}\left\{\exp\left[-\frac{(x-m^{(q)}(0))^2}{2\alpha}\right]+\exp\left[-\frac{(x+m^{(q)}(0))^2}{2\alpha}\right]\right\}. \tag{13}$$

Independence of $\xi_i^{(q)}\xi_i^{(p)}\xi_j^{(p)}$ and $w_j^{p,q}(t=0)$ and the symmetry of (13) provide the same probability density for variable $\xi_i^{(q)}\xi_i^{(p)}\xi_j^{(p)}w_j^{p,q}(t=0)$. Then one gets for variable (10) in the 'α'-lim the following conditional expectation:

$$\boldsymbol{E}(\phi_{i,j}^{p,q}(t=1)|\xi_i^{(q)}s_i(0))=\frac{2f_{w(0)}(x=0)}{2C}\,\xi_i^{(q)}s_i(0)+\mathrm{o}\left(\frac{1}{C}\right) \tag{14}$$

and variance:

$$\operatorname{Var}(\phi_{i,j}^{p,q}(t=1)|\xi_i^{(q)}s_i(0))=1-\left(\frac{2f_{w(0)}(x=0)}{2C}\,\xi_i^{(q)}s_i(0)\right)^2+\mathrm{o}\left(\frac{1}{C^2}\right). \tag{15}$$

Using (14) one obtains:

$$\text{'}\alpha\text{'-lim } \boldsymbol{E}(v_{i,I_i}^{q}(t=1)|\xi_i^{(q)}s_i(0))=2\alpha f_{w(0)}(0)\xi_i^{(q)}s_i(0). \tag{16}$$

According to (10) and (12), for any fixed realization $\xi_i s_i(t=0)$ the sequence

$$\left\{\frac{1}{\sqrt{2C}}\sum_{p(\neq q)}^{M}\phi_{i,j}^{p,q}(t)\right\}_{j\in I_i}$$

is IIDRV. Hence, we can apply to this sequence (for a fixed $\xi_i s_i(t=0)$) the CLT. Then, using (14)–(16) and (5), we get

$$\text{'}\alpha\text{'-lim } v_{i,I_i}^{q}(t=1)\stackrel{d}{=}2\alpha f_{w(0)}(0)\xi_i^{(q)}s_i(0)+\sqrt{\alpha}\mathcal{N}(0,1) \tag{17}$$

where $\xi_i^{(q)} s_i(0)$ and $\mathcal{N}(0,1)$ are independent. Hence (cf (4))

$$\text{`}\alpha\text{'-lim } \xi_i^{(q)} s_i(t=2) \overset{d}{=} \text{sign}[m^{(q)}(t=1) + 2\alpha f_{w(0)}(0)\xi_i^{(q)} s_i(0) + \sqrt{\alpha}\mathcal{N}(0,1)] \tag{18}$$

and consequently

$$\begin{aligned} m^{(q)}(t=2) &= \boldsymbol{E}(\text{`}\alpha\text{'-lim } \xi_i^{(q)} s_i(t=2)) \\ &= \sum_{\sigma=\pm 1} \frac{1+\sigma m^{(q)}(t=0)}{2} \operatorname{erf}\left[\frac{m^{(q)}(t=1)+\sigma g(t=1)}{\sqrt{\alpha}}\right] \end{aligned} \tag{19}$$

where

$$g(t=1) = 2\alpha f_{w(0)}(0). \tag{20}$$

Remark. If we put in (19) $g(t=0)=0$, then we return to the DGZ formula, which has a universal form (3) for any $t \geqslant 0$. Thus, it is this term which exactly takes into account feedback for symmetric extremely diluted networks in the 'α'-lim.

Using the same line of reasoning as above, we can proceed to $t=3$. Then

$$\xi_i^{(q)} s_i(t=3) = \text{sign}\left[\frac{1}{2C}\sum_{j\in I_i} \xi_j^{(q)} s_j(t=2) + v_{i,I_i}^{(q)}(t=2)\right] \tag{21}$$

where (see (5) and (10))

$$v_{i,I_i}^{(q)}(t=2) = \frac{1}{2C}\sum_{p(\neq q)}^{M}\sum_{j\in I_i} \text{sign}\left[\frac{1}{2C}\xi_i^{(q)} s_i(t=1) + \xi_i^{(q)}\xi_i^{(p)}\xi_j^{(p)} w_j^{p,q}(t=1)\right]. \tag{22}$$

From representation (5), (11) and the limit (17) one gets

$$\text{`}\alpha\text{'-lim } w_j^{p,q}(t=1) \overset{d}{=} \xi_j^{(q)} m^{(q)}(t=1) + s_j(t=0) g(t=1) + \sqrt{\alpha}\mathcal{N}(0,1). \tag{23}$$

The density of the probability distribution for (23) (cf (12) and (13)) has the form

$$f_{w(1)}(x) = \frac{1}{\sqrt{8\pi\alpha}} \sum_{\sigma_1,\sigma_2=\pm 1} \frac{1+\sigma_1\sigma_2 m^{(q)}(0)}{2} \exp\left[-\frac{(x+\sigma_1 m^{(q)}(t=1)+\sigma_2 g(t=1))^2}{2\alpha}\right]. \tag{24}$$

Then by the same calculation as in (14)-(17), we get for (22):

$$\text{`}\alpha\text{'-lim } v_{i,I_i}^{q}(t=2) \overset{d}{=} g(t=2)\xi_i^{(q)} s_i(t=1) + \sqrt{\alpha}\mathcal{N}(0,1) \tag{25}$$

where $g(t=2) = 2\alpha f_{w(1)}(0)$. Consequently, for (21) we obtain

$$\text{`}\alpha\text{'-lim } \xi_i^{(q)} s_i(t=3) \overset{d}{=} \text{sign}[m^{(q)}(t=2) + \xi_i^{(q)} s_i(t=1) g(t=2) + \sqrt{\alpha}\mathcal{N}(0,1)]. \tag{26}$$

Finally, from (24) and (26) one gets for the main overlap at $t=3$ the following representation (cf (19), (20)):

$$m^{(q)}(t=3) = \sum_{\sigma=\pm 1} \frac{1+\sigma m^{(q)}(t=1)}{2} \operatorname{erf}\left[\frac{m^{(q)}(t=2)+\sigma g(t=2)}{\sqrt{\alpha}}\right] \tag{27}$$

$$g(t=2) = \sqrt{\frac{2\alpha}{\pi}} \sum_{\sigma=\pm 1} \frac{1+\sigma m^{(q)}(t=0)}{2} \exp\left[-\frac{(m^{(q)}(t=1)+\sigma g(t=1))^2}{2\alpha}\right].$$

Summarizing (21)-(28) we conclude that the next step in dynamics of the main overlap can be calculated by the straightforward repetition of the above procedure.

Therefore, the chain of the coupled equations for evolution of the main overlap $m^{(q)}(t=n)$, $n=1,2,\ldots$, gets the form:

$$m^{(q)}(t=n)=\sum_{\sigma=\pm 1}\frac{1+\sigma m^{(q)}(t=n-2)}{2}\operatorname{erf}\left[\frac{m^{(q)}(t=n-1)+\sigma g(t=n-1)}{\sqrt{\alpha}}\right] \tag{28}$$

$$g(t=n)=\sqrt{\frac{\alpha}{\pi}}\sum_{\sigma=\pm 1}\frac{1+\sigma m^{(q)}(t=n-2)}{\sqrt{2}}\exp\left[-\frac{(m^{(q)}(t=n-1)+\sigma g(t=n-1))^2}{2\alpha}\right]$$

with the following initial conditions:

$$m^{(p)}(t=0)=m^{(q)}(0)\delta_{p,q} \qquad m^{(p)}(t=-1)=0 \qquad p=1,2,\ldots \qquad g(t=0)=0. \tag{29}$$

For the saturation limit we get $\alpha_c \approx 1.54$.

To derive the corresponding chain for $m^{(q)}(t)$ in the case of non-zero temperature ($\theta \neq 0$) one can use the following remark. Dynamics (1) is equivalent to stochastic equation

$$s_i(t+1)=\operatorname{sign}\left[\sum_j J_{ij}s_j(t)+\eta_i(t)\right]. \tag{30}$$

Here $\{\eta_i(t)\}_{i=1,\,t=1}^{\infty,\,\infty}$ are IIDRV with distribution $\Pr\{\eta_i(t)\leqslant x\}=\frac{1}{2}(1+\tanh \beta x)$, $\beta=\theta^{-1}$, which reproduce a heat-bath temperature noise for temperature θ. As a consequence one gets additional noisy term η in the right-hand side of (23) and (25). This means that the equations have to be modified as follows:

$$m^{(q)}(t=n)=\sum_{\sigma=\pm 1}\frac{1+\sigma m^{(q)}(t=n-2)}{2\sqrt{2\pi\alpha}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{y^2}{2\alpha}\right\}$$
$$\times\tanh \beta(m^{(q)}(t=n-1)+\sigma g(t=n-1)+y)\,\mathrm{d}y$$

$$g(t=n)=\sqrt{\frac{2\alpha}{\pi}}\sum_{\sigma=\pm 1}\frac{1+\sigma m^{(q)}(t=n-2)}{2}\int_{-\infty}^{+\infty}\exp\left\{-\frac{y^2}{2\alpha}\right\} \tag{31}$$
$$\times\frac{\beta}{2\cosh^2 \beta(y-m^{(q)}(t=n-1)-\sigma g(t=n-1))}\,\mathrm{d}y$$

with initial conditions (29).

In this letter we have demonstrated that the symmetric version of the DGZ model in 'α'-lim can be solved exactly. In the case of parallel dynamics $D_t^{(\theta)}$ we get explicit equations for evolution of the main overlap. We have shown that this evolution is described by the chain of the coupled equations, see (28) and (32). One of the interesting features of this dynamics is its formal two-step memory which was anticipated in [6].

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